Computer program for the analytical solution of inviscid Stokes standing waves of order (*kH*)5

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# Abstract

In this short paper, a previously published 5th order standing wave theory based on perturbation technique is reproved using an open source computer algebra system. The expansion, based on the small parameter *kH* which wave height *H* multiplied by wave number *k*, is derived for inviscid and incompressible standing waves.

# Introduction

Standing waves are an important problem in waves and fluid mechanics because of their various applications. They can occur in lakes under the forcing of wind, in water tankers under the influence of external forcing and in fuel tankers of vehicles. It is important to determine maximum wave amplitude and the period of motion to be able to provide means of resistance against fluid forces.

[Sobey (2009)](#_ENREF_5) gave a 5th order analytical solution for Stokes wave theory which is based on inviscid and irrotational potential function. The solution is performed by writing Fourier series for potential function *ϕ* as well as free surface position *η*, with respect to a dimensionless and small variable *ε* (wave height divided by wavelength) up to 5th order. The potential function already satisfies the Laplace equation for continuity of fluid, consequently, the coefficients of the Fourier series were obtained by substitution in the boundary conditions. [Sobey (2009)](#_ENREF_5) provided coefficients in the appendix of their paper. Due to some typographical errors in that version, an errata paper was published to correct values of some of the coefficients in [Sobey (2012)](#_ENREF_4). Due to tedious algebra involved in obtaining the coefficients, computer algebra system was used by [Sobey (2009)](#_ENREF_5) to carry out the symbolic manipulations. The work is a continuation of work [Tadjbakhsh and Keller (1960)](#_ENREF_7) who provided standing wave solutions up to 3rd order. [Sobey (2012)](#_ENREF_6) extended the theory until 20th order approximation, although the coefficients were obtained by meshing the grid and solving a nonlinear optimization numerical problem.

In general, research papers are expected to provide novel results and methods to advance the state of science. However, it is also beneficial to reproduce the works of previous papers in order to obtain an independent verification of their results, and also to increase the trustworthiness of the paper in the eyes of the scientific community. In this short paper, we have tried to reproduce the analytical work of [Sobey (2009)](#_ENREF_5). The technical details which are needed for implementation of the solution in the computer algebra system are included. The computer algebra software notebook file as well as the computer code for the standing wave calculations are provided as electronic supplementary files to facilitate the application of the method.

# Governing equations and boundary conditions

The governing equation and boundary conditions, according to [Sobey (2009)](#_ENREF_5) are:

|  |  |
| --- | --- |
| (Continuity),  (Kinematic free surface boundary condition),  (Dynamic free surface boundary condition),  (Impermeable bottom and side walls),  (Initial conditions),  and  (Periodic lateral boundary condition)  and  (Wave height constraint)  (Mean water level (MWL) is at *z* = 0) | (1) |

Wherein *ϕ* = potential function, *η* = free surface position, *x* and *z* = horizontal and vertical coordinates, *t* = time, *u* and *w* = horizontal and vertical velocity components, *g* = acceleration of gravity, *B̅* = Bernoulli's constant, *L* = wavelength, *h* = mean water depth, *η*0 = initial disturbance of water surface, *k* = 2*π*/*L* = wave number, *ω* = 2*π*/*T* = angular frequency, *T* = period, *H* = wave height, *fK* and *fD* = errors of kinematic and dynamic boundary conditions, and subscripts represent differentiation.

# Perturbation expansions for *ϕ*, *η*, *ω*, *B̅*

The following Fourier series are written for the variables of interest as a perturbation expansion with respect to a small parameter, which is chosen to be *ε* = *kH*/2 in [Sobey (2009)](#_ENREF_5). The expansions are given by:

|  |  |
| --- | --- |
| , , | (2) |

Here *Aijm*, *bijm*, *Ci* and *Di* = dimensionless coefficients for *ϕ*, *η*, *ω* and *B̅*, respectively; *N* = summation bound for order of expression, *Nη* = order of calculating *η*, *Nω* = order of calculating *ω*; and *i*, *j*, and *m* are counters. Other quantities can be obtained by direct differentiation of *ϕ* and *η*. As a result ([Sobey 2009](#_ENREF_5)) expressions for *ϕt*, *u*, *w*, *ηx* and *ηt* can be obtained as well.

For simplifying the *ϕ* calculated at *z* = *η*, a trigonometric expansion is applied on cosh function and then the Taylor series expansion with respect to *z* = 0 is applied based on [Sobey (2009)](#_ENREF_5).

|  |  |
| --- | --- |
|  | (3) |

Multiple angle terms tanh *jkh* can be expressed in terms of single angle. By defining the dimensionless parameter *q* = tanh *kh*, the following formulas can be obtained:

|  |  |
| --- | --- |
|  | (4) |

The patterns show that tanh *jkh* is a rational function of *q* and *j*, denoted by .

# Nondimensionalization

The following nondimensionalization is done in order to simplify equations:

|  |  |
| --- | --- |
|  | (5) |

The dimensionless quantities are demonstrated with hats. Consequently, the dimensionless expressions for *ϕ* and *η* along with the rest of variables (without showing the hat signs) are given by:

|  |  |
| --- | --- |
| ,  , | (6) |
|  | (7) |

Here *Nt* = order of calculating the Taylor series, *ñ* = counter for *Nt*, *i'* = counter for *Nω*, and *i"* = counter for *Nη*. The boundary conditions in dimensionless form are:

|  |  |
| --- | --- |
| (periodic boundary conditions)  (wave height constraint)  (kinematic boundary condition)  (dynamic boundary condition) | (8) |

# Extra conditions for coefficients *A* and *b*

From MWL condition, we can obtain that *bijm* = 0 for *j* = 0. Additional conditions can be used for speeding up calculations, which are: *Aijm* = *bijm* = 0 for *i* + *j* = odd, or *i* + *m* = odd and *j* + *m* = odd , where the last condition was given by [Sobey (2009)](#_ENREF_5). In addition, *Ci* = 0 for *i* = even and *Di* = 0 for *i* = odd. Employing the additional conditions is not necessary for finding the solution.

# Simplification of sine and cosine functions

The boundary conditions for order *N*, with *N* = even while eliminating all orders other than *εN* can be simplified to:

|  |  |
| --- | --- |
|  | (9) |

Here *β* and *γ* are coefficients which are functions of *q*, and *γ* is a function of previous order coefficients. The values of *DN*, *AN*,*j*=0,*m* and *bN*,*j*,*m*=0 are directly obtained from *fD*. The remaining pairs of *AN*,*j*,*m* and *bN*,*j*,*m* are calculated by solving 2×2 systems from *fK* and *fD*. For *N* = even:

|  |  |
| --- | --- |
|  | (10) |

Pairs of *AN*,*j*,*m* and *BN*,*j*,*m* for *j* and *m* not both equal to zero are obtained from solving 2×2 systems from *fK* and *fD*. *BN*,1,1 is subsequently found from wave height constraint, and the 2×2 equation for *AN*,1,1 and *CN* is solved in the end. Each term in the initial expression for *fK* and *dD* may be in terms of multiplication and powers of several angles (such as sin2*x* sin 3*x* cos2 *t*), which need to be converted to multiple angle expressions in Eqs. (9) and (10) either by simplification (as explained in [Sobey (2009)](#_ENREF_5)), or by using orthogonality property of Fourier series. For example, the coefficient of cos *jx* sin *mt* in *fD* is equal to ; The coefficient of cos *mt* is equal to  and the remaining constant coefficient is .

# Selection of *N*, *Nω*, *Nt* and *Nη*

When working with a given degree of accuracy of *N̅*, the values of *N*, *Nω* and *Nη* (upper bounds in calculating *ϕ*, *ω* and *η*), as well as the Taylor series power *Nt* should be chosen as small as possible while still producing results of order *N̅* to save computation resources. For *η*, this means that *N* = *N̅*. For *ηt* we have:

|  |  |
| --- | --- |
|  | (11) |

Where ~ means that the two terms are of the same order. For *ηt*, we must select *N* = *N̅* and *Nω* = *N* − *i* + 1. This will ensure that the highest order term will be of order *O*(*εN̅*) at most. For *ϕ*, *u* and *w*, the multinomial expansion involving the expansion for *η* and the powers of the Taylor series should be expanded, resulting in:

|  |  |
| --- | --- |
|  | (12) |

For *ϕt*, as before we have *Nω* = *N̅* − *i* + 1. For the nonlinear terms *uηx*, *u*2 and *w*2, the binomial expansion is written and decision is made to retain each term. For instance for *uηx* we can consider (*u*1*ε* + *u*2 *ε*2 + *u*3 *ε*3 + … + *ui*1 *εi*1 + …+ *uN1* *εN1* )∙(*ηx*1 *ε* + *ηx*2 *ε*2 + *ηx*3 *ε*3 + … + *ηxi*1 *εi*1 + … + *ηxN2* *εN2*) = *u*1 *ηx*1 *ε*2 + ( *u*1 *ηx*2 + *u*2 *ηx*1 ) *ε*3 + …; and the expansion is continued until obtaining *εN̅*.

# Solution of the system

For the first order solution, we can use *N̅* = 1. The nonlinear terms formed from multiplication need not be included. The wave height constraint becomes 2*ε*(*b*1,1,0+*b*1,1,1−1)=0; the kinematic boundary condition becomes *fK* = *ε* cos*x* sin*t* (*b*1,1,1*C*1+*A*1,1,1)=0, and the dynamic boundary condition becomes *fD* = *ε*(−*D*1+*b*1,1,0 cos*x* + *A*1,0,1 *C*1 cos*t* + (*A*1,1,1 *C*1 + *B*1,1,1)cos*t* cos*x*) = 0. These nonlinear equations have two solutions, but since *ω* > 0, therefore *D*1 = *b*1,1,0 = *A*1,0,1 = 0, ,  and *b*1,1,1 = 1. For higher orders, the resulting equations are linear, as pointed out by [Sobey (2009)](#_ENREF_5).

# Iteration for calculating wavelength, *L*

The method employed by [Sobey (2009)](#_ENREF_5) starts with a given value of water depth *h*, wave height *H* and wavelength *L*, then the wave period *T* is explicitly obtained without the need for iteration. If *H*, *h* and *T* are specified, *L* is found from iteration as outlined below:

1- Input *g*, *h*, *H* and *T*

2- Guess first value of *L*. By using the first order approximation for *ω*, we have . If the deep water approximation is used, then , leading to . If shallow water approximation is used, then , leading to . The average of these two values can be used as the first guess for *L*.

3- Calculate *ω* = 2*π*/*T*, *k* = 2*π*/*L*, *ε* = *kH*/2 and *q* = tanh *kh*

4- Calculate *ωcalc* from the perturbation series solution, and obtain *Tcalc* = 2*π* / *ωcalc*

5- Update *L* using the formula *Lnew* = *Lold* + (*T* − *Tcalc*) × 10

6- Repeat steps 3-5 about 20 times until *T* − *Tcalc* becomes small enough.

# Checking the order of the method

In order to numerically check the order of the method, [Sobey (2009)](#_ENREF_5) suggests calculating residual of kinematic and dynamic free surface boundary conditions *fk* and *fD* for different wave heights by using sample values of *h* = 100 m, *T* = 10 s, *H* = 0.5 m, 1 m, 5 m and 10 m. The residual of *fK* is computed on a 50×50 grid via:

|  |  |
| --- | --- |
|  | (13) |

The plot of *fK* and *fD* vs. *ε* is shown in Figure 1, confirming that the expressions of [Sobey (2009)](#_ENREF_5) are *O*(*ε*5).

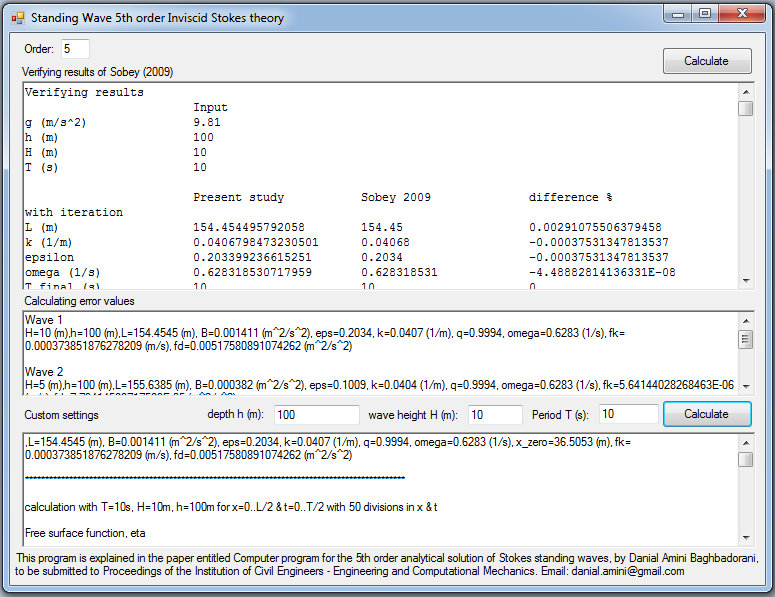
**Figure 1.** Plot of *fK* and *fD* vs. *ε*

Using the proposed formula, the wave profiles for a specific case with *ε* ≈ 0.2 were plotted in Figure 2, similar to graphs of [Sobey (2009)](#_ENREF_5). It is observed that the location of zero height of the wave does not exactly fall at *x* = *L*/4 point, but always towards the peak side, because in nonlinear and steep waves, it is well known that the peaks are steep and the troughs are flat. The length of locus of *z* = 0 is equal to 0.027 *L* for the particular example. Two pathlines are also drawn in Figure 2 which show the path of the particles in standing waves. The path clearly shows that any particle on the free surface stays on the free surface, and only moves along the free surface but cannot detach itself from it. Another notable point is that unlike traveling waves, the pathlines are not closed loops, but are curved lines. If the plots are drawn from *T*/2 until *T*, it will be seen that the particle will traverse back along the exact same pathline to its initial position. Also the particle which resides at *z* = *η* and *x* = 0 moves along a vertical pathline, always attached to both the sidewall and the free surface. The theory of [Sobey (2009)](#_ENREF_5) is only valid for the first mode of standing waves, meaning that the dimension of the container is half the wavelength.

**Figure 2.** Standing wave profile for different instances for perturbation series up to order *N*=5

# Supplementary materials

The computer algebra system used for symbolic computations was [SageMath (2018)](#_ENREF_3), which is an open source and free software, and it can be downloaded from http://www.sagemath.org/. Sage notebooks have \*.ipynb file extension. A simple program was made for calculating wave profiles using Visual Studio Express, a free software provided by [Microsoft Coorporation (2018)](#_ENREF_2), downloadable from https://www.visualstudio.com/vs/visual-studio-express/. The files are available from the journal website and also from the Github repository of [Amini Baghbadorani (2018)](#_ENREF_1).



**Figure 3.** Wave calculator software

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